KINEMATICS OF FLUID FLOW

Fluid kinematics refers to the features of a fluid in motion. It only deals with the motion of fluid particles without taking into account the forces causing the motion. Considerations of velocity, acceleration, flow rate, nature of flow and flow visualization are taken up under fluid kinematics.

A fluid motion can be analyzed by one of the two alternative approaches, called **Lagrangian** and **Eulerian**.

In Lagrangian approach, a particle or a fluid element is identified and followed during the course of its motion with time as demonstrated in Fig.1

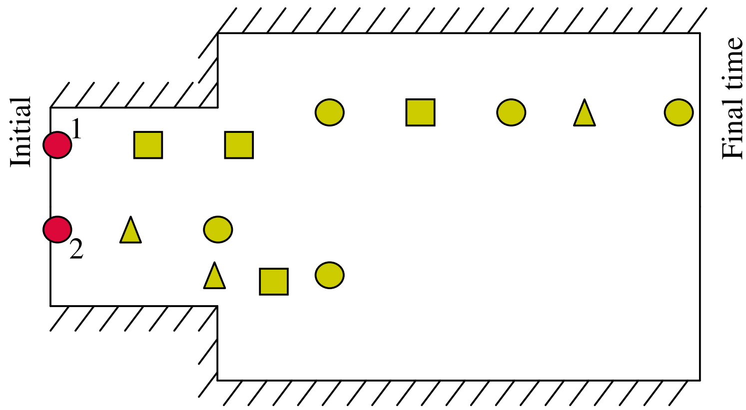


Fig. 1. Lagrangian Approach (Study of each particle with time)

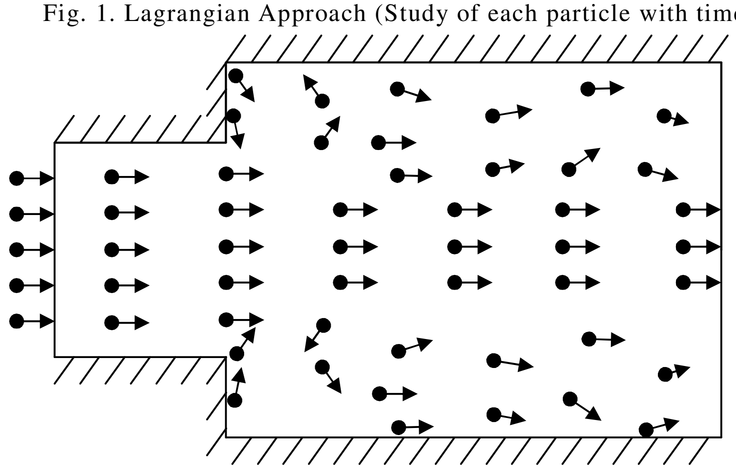


Fig. 2. Eulerian Approach (Study at fixed station in space)

Eg: To know the attributes of a vehicle to be purchased, you can follow the specific vehicle in the traffic flow all along its path over a period of time.

Difficulty in tracing a fluid particle (s) makes it nearly impossible to apply the Lagrangian approach. The alternative approach, called Eulerian approach consists of observing the fluid by setting up fixed stations (sections) in the flow field (Fig. 2).

Motion of the fluid is specified by velocity components as functions of space and time. This is considerably easier than the previous approach and is followed in Fluid Mechanics.

Eg: Observing the variation of flow properties in a channel like velocity, depth etc, at a section.

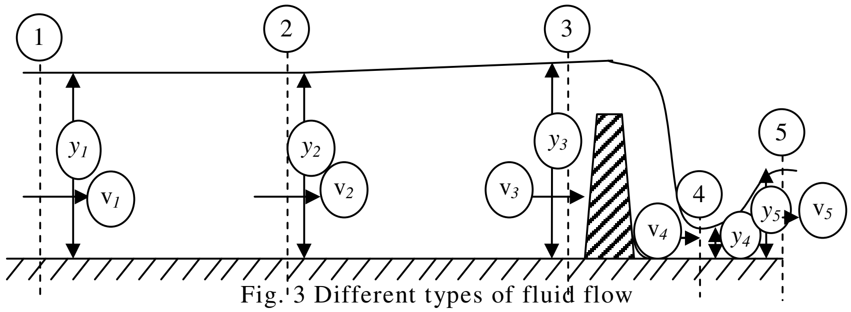


Fig. 3 Different types of fluid flow

Classification of Flows:

1. Steady and unsteady flows:

A flow is said to be steady if the properties (P) of the fluid and flow do not change

$$\frac{\partial}{\partial t}(P) = 0$$

with time (t) at any section or point in a fluid flow.

A flow is said to be unsteady if the properties (P) of the fluid and flow change with

$$\frac{\partial}{\partial t}(P) \neq 0$$

time (t) at any section or point in a fluid flow.
Eg: Flow observed at a dam section during rainy season, wherein, there will be lot of inflow with which the flow properties like depth, velocity etc.. will change at the dam section over a period of time representing it as unsteady flow.

2. Uniform and non-uniform flows:

A flow is said to be uniform if the properties (P) of the fluid and flow do not change (with direction) over a length of flow considered along the flow at any instant.

$$\frac{\partial}{\partial x}(P) = 0 \qquad \qquad \frac{\partial}{\partial x}(P) \neq 0$$

A flow is said to be ***non-uniform*** if the properties (P) of the fluid and flow change (with direction) over a length of flow considered along the flow at any instant.
Eg: Flow observed at any instant, at the dam section during rainy season, wherein, the flow varies from the top of the overflow section to the foot of the dam and the flow properties like depth, velocity etc., will change at the dam section at any instant between two sections, representing it as non-uniform flow.
Consider a fluid flow as shown above in a channel. The flow is said to be steady at sections 1 and 2 as the flow does not change with respect to time at the respective sections (y_1=y_2 *and* v_1=v_2).
The flow between sections 1 and 2 is said to be uniform as the properties does not change between the sections at any instant (y_1=y_2 and v_1= v_2).
The flow between sections 2 and 3 is said to be non-uniform flow as the properties vary over the length between the sections.

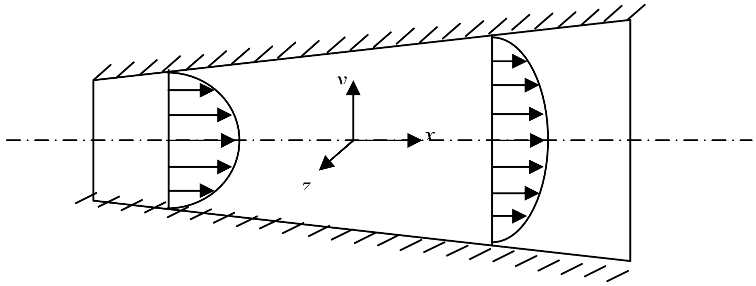


Fig. 4 c) Three dimensional flow

Non-uniform flow can be further classified as ***Gradually varied flow*** and ***Rapidly varied flow***. As the name itself indicates, ***Gradually varied flow*** is a non-uniform flow wherein the flow/fluid properties vary gradually over a long length (Eg: between sections 2 and 3).

Rapidly varied flow is a non-uniform flow wherein the flow/fluid properties vary rapidly within a very short distance. (Eg: between sections 4 and 5).

Combination of steady and unsteady flows and uniform and non-uniform flows can be classified as ***steady-uniform flow*** (Sections 1 and 2)***, unsteady-uniform flow, steady-non-uniform flow*** (Sections 2 and 3) and ***unsteady-non-uniform flow*** (Sections 4 and 5)**.**

3. *One, two and three dimensional flows:*

Flow is said to be ***one-dimensional*** if the properties vary only along one axis / direction and will be constant with respect to other two directions of a three-

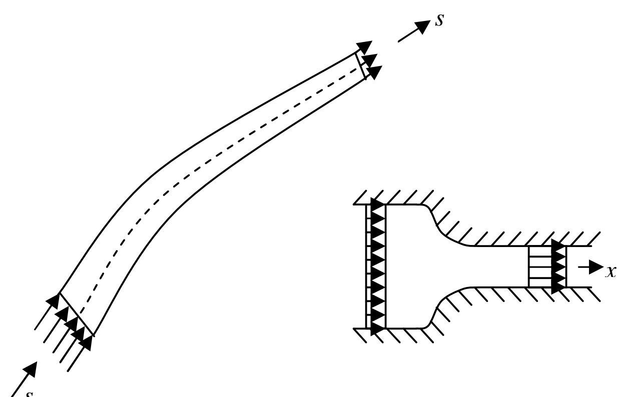


Fig. 4 a) One dimensional flow

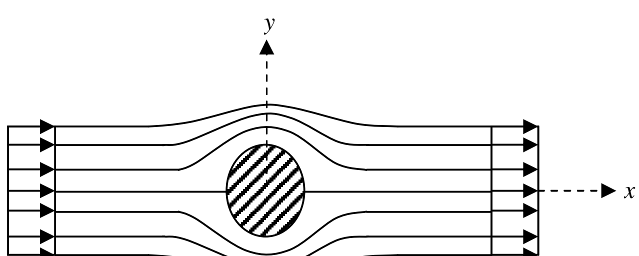


Fig. 4 b) Two dimensional flow

dimensional axis system.

Flow is said to be ***two-dimensional*** if the properties vary only along two axes / directions and will be constant with respect to other direction of a three-dimensional axis system.

Flow is said to be ***three-dimensional*** if the properties vary along all the axes / directions of a three-dimensional axis system.

4. *Laminar and Turbulent flows:*

When the flow occurs like sheets or laminates and the fluid elements flowing in a layer does not mix with other layers, then the flow is said to be laminar. The Reynolds number (R_e) for the flow will be less than 2000.

$$R_e = \frac{\rho v D}{\mu}$$

When the flow velocity increases, the sheet like flow gets mixed up and the fluid elements mix with other layers there by causing turbulence. There will be eddy currents generated and flow reversal takes place. This flow is said to be Turbulent.

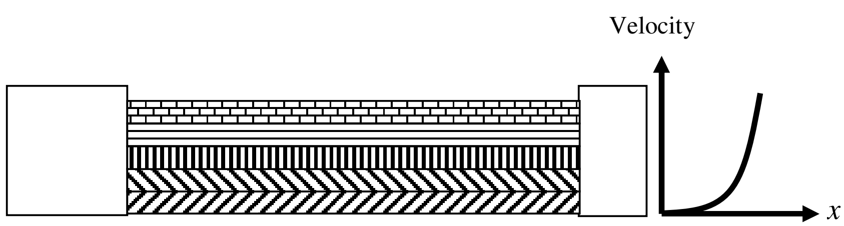


Fig. 5 Laminar flow

The Reynolds number for the flow will be greater than 4000.
For flows with Reynolds number between 2000 to 4000 is said to be transition flow.

5. *Compressible and Incompressible flows:*

Flow is said to be ***Incompressible*** if the fluid density does not change (constant) along the flow direction and is ***Compressible*** if the fluid density varies along the flow direction

ρ = Constant (incompressible) and $\rho \neq$ Constant (compressible)

6. *Rotational and Irrotational flows:*

Flow is said to be ***Rotational*** if the fluid elements does not rotate about their own axis as they move along the flow and is ***Rotational*** if the fluid elements rotate along their axis as they move along the flow direction

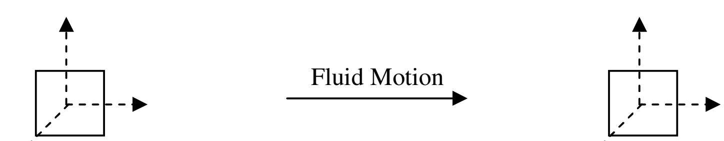


Fig. 6 a)Irrotational flow

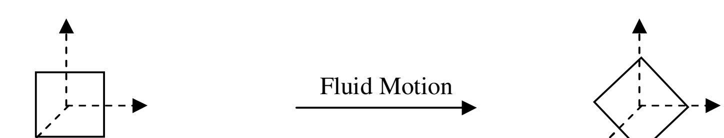


Fig.6 b)Rotational flow

7. Critical, Sub-critical and Super-critical flows:

Froude's Number

It is the ratio of the inertia forces to gravity forces and mathematically

$F_e = \frac{V}{\sqrt{gd}}$, where F_e=Froudes No, V is the flow velocity and d is the hydraulic mean depth given by $d = \frac{A}{T}$, A is the flow cross-sectional area and T is the top width.

If the Foude's number is ONE, the flow is ***critical***, Less than ONE, ***Sub-critical*** and Greater than ONE, ***Super-critical.***

Rate of flow or Discharge (Q):

Rate of flow or discharge is said to be the quantity of fluid flowing per second across a section of a flow. Rate of flow can be expressed as mass rate of flow or volume rate of flow. Accordingly

Mass rate of flow = Mass of fluid flowing across a section / time

Rate of flow = Volume of fluid flowing across a section / time

Types of lines

Path Line: It is the path traced by a fluid particle over a period of time during its motion along the fluid flow.

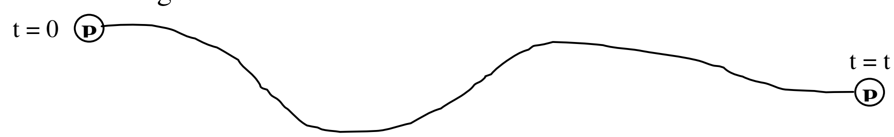
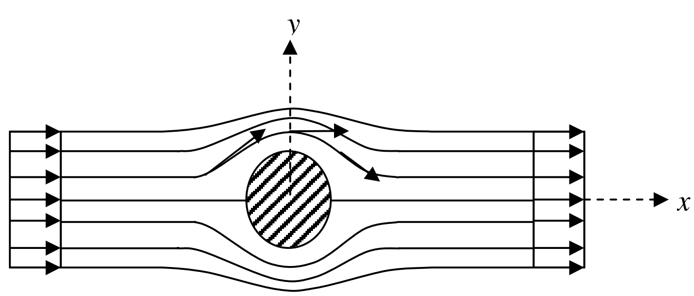


Fig. 7 Path line

Eg: Path traced by an ant coming out from its dwelling

Stream Lines

It is an imaginary line such that when a tangent is drawn at any point it gives the



velocity of the fluid particle at that point and at that instant.

Fig. 8 Stream lines

Eg: Path traced by the flow when an obstruction like, a sphere or a stick is kept during its motion. The flow breaks up before the obstruction and joins after it crosses it.

Streak lines

It is that imaginary line that connects all the fluid particles that has gone through a point/section over a period of time in a fluid motion.

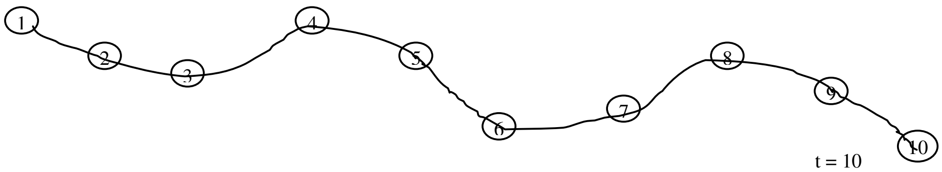


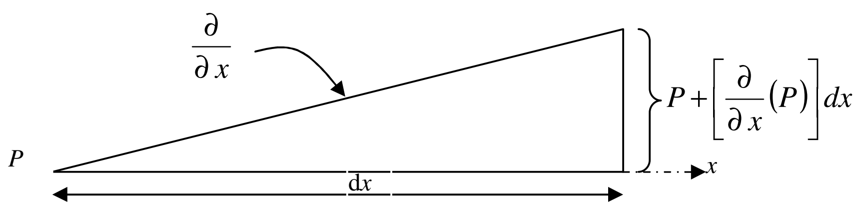
Fig. 9 Streak lines

Stream tube:

It is an imaginary tube formed by stream line on its surface such that the flow only enters the tube from one side and leaves it on the other side only. No flow takes place across the stream tube. This concept will help in the analysis of fluid motion.

Variation of a Property along any given direction

If P is a Property at any point, then the property at any other location along x



direction at a distance dx is given by

Fig. 12 Variation of a property along x direction

New Property = Old Property + slope x Old Property x distance

Continuity Equation

The derivation is based on the concept of Law of conservation of mass.

Statement: The flow of fluid in a continuous flow across a section is always a constant.

Consider an enlarging section in a fluid flow of fluid density γ. Consider two sections 1 and 2 as shown in Fig. Let the sectional properties be as under
A_1 and A_2= Cross-sectional area, V_1 and V_2= Average flow velocity and
ρ_1 and ρ_2 = Fluid density at Sec 1 and 2 respectively
dt is the time taken for the fluid to cover a distance ***dx***
The mass of fluid flowing across section 1-1 is given by
m_1 = Density at section 1* x *volume of fluid that has crossed section 1
$\rho_1 \times A_1 \times dx$
Mass rate of fluid flowing across section 1-1 is given by
m_1/dt = (Density at sec 1* x *volume of fluid that has crossed sec 1)/dt
$\rho_1 \times A_1 \times dx/dt = \rho_1 A_1 V_1$ …(01)
Similarly Mass rate of fluid flowing across section 2-2 is given by
$m_2/dt = \rho_2 \times A_2 \times dx/dt = \rho_2 A_2 V_2$ …(02)

From law of conservation of mass, mass can neither be created nor destroyed.

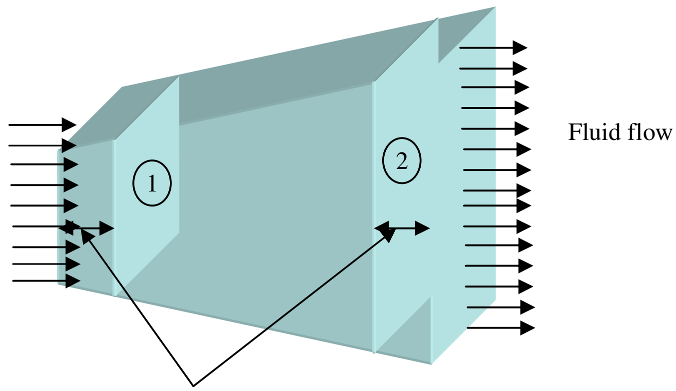


Fig. 13 Fluid flow through a control volume

Hence from Eqs. 1 and 2, we get

$\rho_1 A_1 V_1 = \rho_2 A_2 V_2$

If the density of the fluid is constant, then the equation reduces to

$A_1 V_1 = A_2 V_2$

The above equation is discharge continuity equation in one dimensional form for an steady, incompressible fluid flow

Continuity Equation in three dimensional or differential forms

Consider a parallelepiped ***ABCDEFGH*** in a fluid flow of density γ as shown in Fig. Let the dimensions of the parallelepiped be ***dx***, ***dy*** and ***dz*** along ***x***, ***y*** and ***z*** directions respectively. Let the velocity components along ***x, y*** and ***z*** be ***u,*** **v** and **w** respectively.

Similarly mass rate of fluid flow leaving the section EFGH along *x* direction is given by

$$M_{x2} = \left[\rho u + \frac{\partial}{\partial x}(\rho u)dx\right]dy\,dz \qquad \ldots(02)$$

Net gain in mass rate of the fluid along the *x* axis is given by the difference between the mass rate of flow entering and leaving the control volume. i.e. Eq. 1 – Eq. 2

$$dM_x = \rho u\,dy\,dz - \left[\rho u + \frac{\partial}{\partial x}(\rho u)dx\right]dy\,dz$$

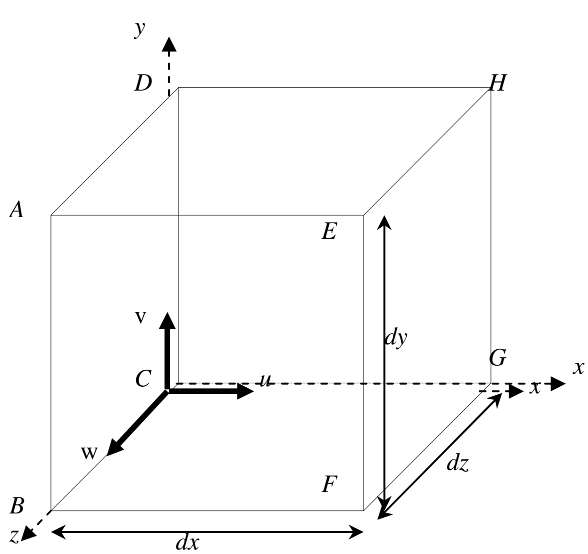


Fig. 14 parallelepiped in a fluid flow

$$dM_x = -\frac{\partial}{\partial x}(\rho u)dx\,dy\,dz \quad \ldots(03)$$

Similarly net gain in mass rate of the fluid along the *y* and *z* axes are given by

$$dM_y = -\frac{\partial}{\partial y}(\rho \mathrm{v})dx\,dy\,dz \quad \ldots(04)$$

$$dM_z = -\frac{\partial}{\partial z}(\rho \mathrm{w})dx\,dy\,dz \quad \ldots(05)$$

Net gain in mass rate of the fluid from all the three axes are given by

$$dM = -\frac{\partial}{\partial x}(\rho u)dx\,dy\,dz - \frac{\partial}{\partial y}(\rho \mathrm{v})dx\,dy\,dz - \frac{\partial}{\partial z}(\rho \mathrm{w})dx\,dy\,dz$$

From law of conservation of Mass, the net gain in mass rate of flow should be zero and hence

$$\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho \mathrm{v}) + \frac{\partial}{\partial z}(\rho \mathrm{w})\right]dx\,dy\,dz = 0$$

or
$$\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho \mathrm{v}) + \frac{\partial}{\partial z}(\rho \mathrm{w})\right] = 0$$

This expression is known as the general Equation of Continuity in three dimensional form or differential form.

If the fluid is incompressible then the density is constant and hence

$$\left[\frac{\partial u}{\partial x} + \frac{\partial \mathrm{v}}{\partial y} + \frac{\partial \mathrm{w}}{\partial z}\right] = 0$$

The continuity equation in two-dimensional form for compressible and incompressible flows are respectively as below

$$\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho \mathrm{v})\right] = 0$$

$$\left[\frac{\partial u}{\partial x} + \frac{\partial \mathrm{v}}{\partial y}\right] = 0$$

Velocity

Velocity of a fluid along any direction can be defined as the rate of change of displacement of the fluid along that direction.

u = dx / dt

Where ***dx*** is the distance traveled by the fluid in time ***dt***.

Velocity of a fluid element is a vector, which is a function of space and time.

Let **V** be the resultant velocity of a fluid along any direction and ***u***, **v** and **w** be the velocity components in ***x, y*** and ***z*** directions respectively.

Mathematically the velocity components can be written as

$u = f(x, y, z, t)$

$\mathrm{v} = f(x, y, z, t)$

$\mathrm{w} = f(x, y, z, t)$

and $V = u\mathbf{i} + \mathrm{v}\mathbf{j} + \mathrm{w}\mathbf{k} = \sqrt{u^2 + \mathrm{v}^2 + \mathrm{w}^2}$

Where $u = (dx/dt)$, $\mathrm{v} = (dy/dt)$ and $\mathrm{w} = (dz/dt)$.

Acceleration

Acceleration of a fluid element along any direction can be defined as the rate of change of velocity of the fluid along that direction.
If a_x , a_y and a_z are the components of acceleration along ***x, y*** and ***z*** directions respectively, they can be mathematically written as
$\boldsymbol{a_x = du/ dt}$.
But $\boldsymbol{u = f\,(x, y, z, t)}$ and hence by chain rule, we can write,

$$a_x = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt} + \frac{\partial u}{\partial t}$$

Similarly

$$a_y = \frac{\partial \text{v}}{\partial x}\frac{dx}{dt} + \frac{\partial \text{v}}{\partial y}\frac{dy}{dt} + \frac{\partial \text{v}}{\partial z}\frac{dz}{dt} + \frac{\partial \text{v}}{\partial t}$$

and
$$a_z = \frac{\partial \text{w}}{\partial x}\frac{dx}{dt} + \frac{\partial \text{w}}{\partial y}\frac{dy}{dt} + \frac{\partial \text{w}}{\partial z}\frac{dz}{dt} + \frac{\partial \text{w}}{\partial t}$$

But $\boldsymbol{u = (dx/dt)}$, $\mathbf{v} = \boldsymbol{(dy/dt)}$ and $\mathbf{w} = \boldsymbol{(dz/dt)}$.

Hence

$$\left.\begin{aligned}
a_x &= \overbrace{u\frac{\partial u}{\partial x} + \text{v}\frac{\partial u}{\partial y} + \text{w}\frac{\partial u}{\partial z}}^{\text{Convective accln}} + \overbrace{\frac{\partial u}{\partial t}}^{\text{Local accln}} \\
a_y &= u\frac{\partial \text{v}}{\partial x} + \text{v}\frac{\partial \text{v}}{\partial y} + \text{w}\frac{\partial \text{v}}{\partial z} + \frac{\partial \text{v}}{\partial t} \\
a_z &= u\frac{\partial \text{w}}{\partial x} + \text{v}\frac{\partial \text{w}}{\partial y} + \text{w}\frac{\partial \text{w}}{\partial z} + \frac{\partial \text{w}}{\partial t}
\end{aligned}\right\}\ \text{Total accln}$$

If A is the resultant acceleration vector, it is given by

$$A = a_x i + a_y j + a_z k$$
$$= \sqrt{a_x^2 + a_y^2 + a_z^2}$$

For steady flow, the local acceleration will be zero

Problems

1. The velocity field in a fluid is given by

$$V_s = (3x+2y)i + (2z+3x^2)j + (2t-3z)k$$

i. What are the velocity components *u,* v, and w ?
ii. Determine the speed at the point (1,1,1).
iii. Determine the speed at time t=2 s at point (0,0,2).

Solution: The velocity components at any point *(x, y, z)* are

$$u = (3x+2y),\ \text{v} = (2z+3x^2),\ \text{w} = (2t-3z)k$$

Substitute *x=1, y=1, z=1 in the above expression*
u = *(3*1+2*1)* = *5,* v = *(2*1+3*1)* = *5,* w = *(2t-3)*
$V^2 = u^2 + \text{v}^2 + \text{w}^2$

$= 5^2 + 5^2 + (2t-3)^2$

$= 4t^2 - 12t + 59$

$V_{(1,1,1)} = \sqrt{(4t^2 - 12t + 59)}$

Substitute $t = 2$ *s, x=0, y=0, z=2 in the above expression for u,* v *and* w

$u = 0$, v $= (4 + 0) = 4$, w $= (4 - 6) = -2$

$V^2_{(0,0,2,2)} = (0 + 15 + 4) = 20$

$V = 4.472$ *units*

2. Calculate the velocity component V given $u = \frac{2}{3}xy^3 - x^2y$ so that the equation of continuity is satisfied (July 2006) (06)

Solution:

The continuity equation for two dimensional incompressible flow is given by

$[(\partial u/\partial x)+(\partial \text{v}/\partial y)] = 0$ ...(01)

$$\frac{\partial u}{\partial x} = \frac{2}{3}y^3 - 2xy$$

From Eq. 01, we have

$$\frac{\partial u}{\partial x} + \frac{\partial \text{v}}{\partial y} = 0 \Rightarrow \frac{2}{3}y^3 - 2xy + \frac{\partial \text{v}}{\partial x} = 0$$

$$\frac{\partial \text{v}}{\partial y} = 2xy - \frac{2}{3}y^3$$

Integrating the above expression with y, we get

$$\text{v} = xy^2 - \frac{1}{6}y^4$$

3. The velocity distribution in a three-dimensional flow is given by:

$u = -x$, v $= 2y$ and w $= (3-z)$. Find the equation of the stream line that passes through point (1,1,1).

Solution: The stream line equation is given by

$$\frac{dx}{u} = \frac{dy}{\text{v}} = \frac{dz}{\text{w}} \text{ or } \frac{dx}{-x} = \frac{dy}{2y} = \frac{dz}{(3-z)}$$

$$\frac{dx}{-x} = \frac{dy}{2y}$$

Integrating we get $-\log_e x = \frac{1}{2}\log_e y + A,$

Where A is an integral constant. Substituting x=1 & y=1, A = 0

$$\therefore \log_e x = \log_e y^{-1/2} \text{ or } x = \frac{1}{\sqrt{y}}$$

Considering the x and z components, $\frac{dx}{-x} = \frac{dz}{(3-z)}$

Integrating we get $-\log_e x = -\log_e(3-z) + B,$

Where B is an integral constant. Substituting x=1 & z=1, $B = \log_e 2$

$$\therefore -\log_e x = -\log_e(3\text{-z}) + \log_e 2 = -\log_e\left(\frac{3-z}{2}\right)$$

$$\text{or } x = \left(\frac{3-z}{2}\right)$$

From Eqs. 1 and 2, the final equation of the stream line that passes through the point (1,1,1) is

$$x = \frac{1}{\sqrt{y}} = \left(\frac{3-z}{2}\right)$$

4. A fluid particle moves in the following flow field starting from the points (2,1,0) at t=0.
Determine the location of the fluid particle at t = 3 s

$$u = \frac{t^2}{2x}, \mathrm{v} = \frac{ty^2}{18}, \mathrm{w} = \frac{z}{2t}$$

Solution

$$u = \frac{dx}{dt} = \frac{t^2}{2x} \text{ or } 2xdx = t^2 dt$$

Integrating we get $x^2 = \frac{t^3}{3} + A$

Where A is an integral constant. Substituting x=2, t=0, A = 4

$$x^2 = \frac{t^3}{3} + 4$$

At t = 3 s, $x^2 = \frac{3^3}{3} + 4$ or $x = \sqrt{13}$

$$\mathrm{v} = \frac{dy}{dt} = \frac{ty^2}{18} \text{ or } \frac{dy}{y^2} = \frac{tdt}{18}$$

Integrating we get

$$-\frac{1}{y} = \frac{t^2}{36} + B$$

Where B is an integral constant.
Substituting y=1, t=0, B = -1

$$\frac{1}{y} = 1 - \frac{t^2}{36}$$

At t = 3 s, $\frac{1}{y} = 1 - \frac{3^2}{36} = \frac{3}{4}$ or $y = \frac{4}{3}$

$$\mathrm{w} = \frac{dz}{dt} = \frac{z}{2t} \text{ or } \frac{2dz}{z} = \frac{dt}{t}$$

Integrating we get

$$2\log_e z = \log_e t + C$$

Where C is an integral constant.
Substituting z=0, t=0, C = 0

$$2\log_e z = \log_e t \text{ or } z^2 = t$$

At t = 3 s, $z^2 = 3$ or $z = \sqrt{3}$

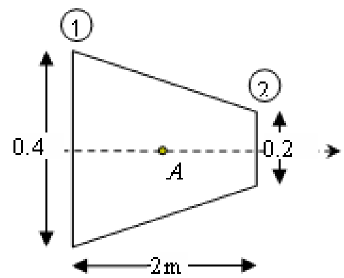
From Eqs. 1, 2 and 3, at the end of 3 seconds the particle is at a point

$$\left(\sqrt{13}, \frac{4}{3}, \sqrt{3}\right)$$

5. The following cases represent the two velocity components, determine the third component of velocity such that they satisfy the continuity equation:
(i) $u = x^2 + y^2 + z^2$ **;** $v = xy^2 - yz^2 + xy$**; *(ii)*** $v = 2y^2$ **;** $w = 2xyz$**.**

Solution:
The continuity equation for incompressible flow is given by
$[(\partial u/\partial x)+(\partial v/\partial y) +(\partial w/\partial z)] = 0$ ...(01)
$u = x^2 + y^2 + z^2;\quad (\partial u/\partial x) = 2x$
$v = xy^2 - yz^2 + xy;\quad (\partial v/\partial y) = 2xy - z^2 + x$
Substituting in Eq. 1, we get
$2x + 2xy - z^2 + z + (\partial w/\partial z) = 0$
Rearranging and integrating the above expression, we get
$\mathbf{w = (-3xz - 2xyz + z^3/3) + f(x,y)}$
Similarly, solution of the second problem
$\mathbf{u = -4xy - x^2y + f(y,z).}$

6. Find the convective acceleration at the middle of a pipe which converges uniformly from 0.4 m to 0.2 m diameter over a length of 2 m. The rate of flow is 20 lps. If the rate of flow changes uniformly from 20 lps to 40 lps in 30 seconds, find the total acceleration at the middle of the pipe at 15th second.
Solution:
$D_1 = 0.4$ m, $D_2 = 0.2$ m, L = 2 m,
Q = 20 lps = 0.02 m^3/s.
$Q_1 = 0.02\ m^3/s$ and $Q_2 = 0.04\ m^3/s$
Case (i)
Flow is one dimensional and hence the velocity components
$v = w = 0$



$\therefore$Convective acceleration = $\mathbf{u(\partial u/\partial x)}$
$\mathbf{A1 = (\pi/4)(D_1^2) = 0.1257\ m^2}$
$\mathbf{A2 = (\pi/4)(D_2^2) = 0.0314\ m^2}$
$\mathbf{u_1 = Q/A_1 = 0.02/0.1257 = 0.159\ m/s}$
and $\mathbf{u_2 = Q/A_2 = 0.04/0.0314 = 0.637\ m/s}$
As the diameter changes uniformly, the velocity will also change uniformly. The velocity u at any distance x from inlet is given by
$\mathbf{u = u_1 + (u_2 - u_1)/(x/L) = 0.159 + 0.2388\ x}$
$\mathbf{(\partial u/\partial x) = 0.2388}$
$\therefore$Convective acceleration = $\mathbf{u(\partial u/\partial x) = (0.159 + 0.2388\ x)\ 0.2388}$
At A**,** $\mathbf{x = 1\ m}$ and hence
(Convective accln) $\mathbf{_{x=1} = 94.99\ mm/s^2}$
Case (ii)
Total acceleration = (convective + local) acceleration at $\mathbf{t = 15}$ **seconds**
Rate of flow $\mathbf{Q_{t=15} = Q_1 + (Q_2 - Q_1)(15/30) = 0.03\ m3/s.}$
$\mathbf{u_1 = Q/A_1 = 0.03/0.1257 = 0.2386\ m/s}$
and $\mathbf{u_2 = Q/A_2 = 0.03/0.0314 = 0.9554\ m/s}$
The velocity u at any distance $\mathbf{x}$ from inlet is given by
$\mathbf{u = u_1 + (u_2 - u_1)/(x/L) = 0.2386 + 0.3584\ x}$
$\mathbf{(\partial u/\partial x) = 0.3584}$
$\therefore$Convective acceleration = $\mathbf{u(\partial u/\partial x) = (0.2386 + 0.3584\ x)\ 0.3584}$

At **A**, $x = 1\ m$ and hence

(Convective accln) $_{x=1}$ **= 0.2139 m/s 2**

Local acceleration

Diameter at **A** is given by $D = D_1 + (D_1 - D_2)/(x/L) = 0.3\ m$

and $A = (\pi/4)(D^2) = 0.0707\ m^2$

When $Q_1 = 0.02\ m^3/s,\ u_1 = 0.02/0.0707 = 0.2829\ m/s$

When $Q_2 = 0.04\ m^3/s,\ u_2 = 0.02/0.0707 = 0.5659\ m/s$

Rate of change of velocity = Change in velocity/time

= (0.5629-0.2829)/30 = 9.43 x 10 -3m/s 2

$\therefore$ **Total acceleration** = $0.2139 + 9.43 \times 10^{-3} = 0.2233\ m/s^2$

Velocity Potential (ϕ)

Velocity Potential ϕ is a scalar function of space and time such that its negative derivative with respect to any direction gives the velocity component in that direction

Thus $\phi = \phi(x,y,z,t)$ and flow is steady then,

$u = -(\partial\phi/\partial x);\ v = -(\partial\phi/\partial y)\ ;\ w = -(\partial\phi/\partial z)$

Continuity equation for a three dimensional fluid flow is given by

$[(\partial u/\partial x)+(\partial v/\partial y) +(\partial w/\partial z)] = 0$

Substituting for ***u***, **v** and **w**, we get

$[(\partial/\partial x)(-\partial\phi/\partial x)+(\partial/\partial y)(-\partial\phi/\partial y) +(\partial/\partial z)(-\partial\phi/\partial z)] = 0$

i.e. $[(\partial^2\phi/\partial x^2)+(\partial^2\phi/\partial y^2)+(\partial^2\phi/\partial z^2)] = 0$

The above equation is known as ***Laplace equation*** in ϕ

For a 2 D flow the above equation reduces to

$$[(\partial^2\phi/\partial x^2)+(\partial^2\phi/\partial y^2)] = 0$$

We know that for an irrotational two dimensional fluid flow, the rotational fluid elements about z axis must be zero.

i.e. $\omega_z = ½\,[(\partial v/\partial x) - (\partial u/\partial y)]$

Substituting for ***u*** and **v**, we get

$\omega_z = ½\,[(\partial/\partial x)(-\partial\phi/\partial y) - (\partial/\partial y)(-\partial\phi/\partial x)]$

For the flow to be irrotational, the above component must be zero

$\omega_z = ½\,[\,(-\partial 2\phi/\partial x\ \partial y) - (-\partial 2\phi/\ \partial y\ \partial x)] = 0$

i.e. $(-\partial^2\phi/\partial x\ \partial y) = (-\partial^2\phi/\ \partial y\ \partial x)$

This is true only when ϕ is a continuous function and exists.

Thus the properties of a velocity potential are:

1. If the velocity potential ϕ exists, then the flow should be irrotational.
2. If the velocity potential ϕ satisfies the *Laplace Equation,* then it represents a possible case of a fluid flow.

Stream Function (ψ)

Stream Function ψ is a scalar function of space and time such that its partial derivative with respect to any direction gives the velocity component at right angles to that direction.

Thus $\psi = \psi(x,y,z,t)$ and flow is steady then,

$u = -(\partial\psi/\partial y);\ v = (\partial\psi/\partial x)$

Continuity equation for a two dimensional fluid flow is given by

$[(\partial u/\partial x)+(\partial v/\partial y)] = 0$

Substituting for u and v, we get

$[(\partial/\partial x)(-\partial \psi/\partial y)+(\partial/\partial y)(\partial \psi/\partial x)] = 0$

i.e. $[(-\partial^2\psi/\partial x\,\partial y) + (\partial^2\psi/\partial y\,\partial x)] = 0$

or $(\partial^2\psi/\partial x\,\partial y) = (\partial^2\psi/\partial y\,\partial x)$

This is true only when ψ is a continuous function.

We know that for an irrotational two dimensional fluid flow, the rotational fluid elements about z axis must be zero.

i.e. $\omega_z = ½ [(\partial v/\partial x) - (\partial u/\partial y)]$

Substituting for u and v, we get

$\omega_z = ½ [(\partial/\partial x)(\partial \psi/\partial x) - (\partial/\partial y)(-\partial \psi/\partial y)]$

For the flow to be irrotational, the above component must be zero

i.e. $[(\partial^2\phi/\partial x^2)+(\partial^2\phi/\partial y^2)] = 0$

The above equation is known as ***Laplace equation*** in ψ

Thus the properties of a Stream function are:

1. If the Stream function ψ exists, then it represents a possible case of a fluid flow.
2. If the Stream function ψ satisfies the ***Laplace Equation***, then the flow should be irrotational.

Equi-potential lines:

It is an imaginary line along which the velocity potential ϕ is a constant

i.e. ϕ = Constant

$\therefore \quad d\phi = 0$

But $\phi = f(x,y)$ for a two dimensional steady flow

$\therefore \quad d\phi = (\partial \phi/\partial x)dx + (\partial \phi/\partial y)dy$

Substituting the values of u and **v,** we get

$d\phi = -u\,dx - v\,dy \Rightarrow 0$

or $u\,dx = -v\,dy$

or $(dy/dx) = -u/v$ … (01)

Where dy/dx is the slope of the equi-potential line.

Line of constant stream function or stream line

It is an imaginary line along which the stream function ψ is a constant

i.e. ψ = Constant

$\therefore \quad d\psi = 0$

But $\psi = f(x,y)$ for a two dimensional steady flow

$\therefore \quad d\psi = (\partial \psi/\partial x)dx + (\partial \psi/\partial y)dy$

Substituting the values of u and **v,** we get

$d\psi = v\,dx - u\,dy \Rightarrow 0$

or $v\,dx = u\,dy$

or $(dy/dx) = v/u$ … (02)

Where dy/dx is the slope of the Stream line.

From Eqs. 1 and 2, we get that the product of the slopes of equi-potential line and stream line is given by ***-1.*** Thus, the equi-potential lines and stream lines are orthogonal to each other at all the points of intersection.

Flow net

A grid obtained by drawing a series of equi-potential lines and stream lines is called a ***Flow net.*** The flow net is an important tool in analysing two dimensional flow irrotational flow problems.

Relationship between Stream function (ψ) and Velocity potential (ϕ)

We know that the velocity components are given by

$u = -(\partial \phi/\partial x)$ $\quad$ $v = -(\partial \phi/\partial y)$

and $u = -(\partial \psi/\partial y)$ $\quad$ $v = (\partial \psi/\partial x)$

Thus $u = -(\partial \phi/\partial x) = -(\partial \psi/\partial y)$ and $v = -(\partial \phi/\partial y) = (\partial \psi/\partial x)$

Hence $(\partial \phi/\partial x) = (\partial \psi/\partial y)$

and $(\partial \phi/\partial y) = -(\partial \psi/\partial x)$

Problems

Jan/Feb 2003

6. In a two dimensional incompressible flow the fluid velocity components are given by

$u = x - 4y$ and $v = -y - 4x$

Where u and v are x and y components of flow velocity. Show that the flow satisfies the continuity equation and obtain the expression for stream function. If the flow is potential, obtain also the expression for the velocity potential. (07)

Solution:

$u = x - 4y$ and $v = -y - 4x$

$(\partial u/\partial x) = 1$ and $(\partial v/\partial y) = -1$

$(\partial u/\partial x) + (\partial v/\partial y) = 1 - 1 = 0.$

Hence it satisfies continuity equation and the flow is continuous and velocity potential exists.

Let ϕ be the velocity potential.

Then $(\partial \phi/\partial x) = -u = -(x - 4y) = -x + 4y$ (1)

and $(\partial \phi/\partial y) = -v = -(-y - 4x) = y + 4x$ (2)

Integrating Eq. 1, we get

$\phi = (-x^2/2) + 4xy + C$ (3)

Where C is an integral constant, which is independent of x and can be a function of y.

Differentiating Eq. 3 w.r.t. y, we get

$(\partial \phi/\partial y) = 0 + 4x + (\partial C/\partial y) \Rightarrow y + 4x$

Hence, we get $(\partial C/\partial y) = y$

Integrating the above expression, we get $C = y^2/2$

Substituting the value of C in Eq. 3, we get the general expression as

$\phi = (-x^2/2) + 4xy + y^2/2$

Stream Function

Let ψ be the velocity potential.

Then $(\partial \psi/\partial x) = v = (-y - 4x) = -y - 4x$ (4)

and $(\partial \psi/\partial y) = u = -(x - 4y) = -x + 4y$ (5)

Integrating Eq. 4, we get

$$\psi = - y\, x - 4\, (x2/2) + K \quad (6)$$

Where K is an integral constant, which is independent of x and can be a function of y.

Differentiating Eq. 6 w.r.t. y, we get

$$(\partial \psi / \partial y) = - x - 0 + (\partial K / \partial y) \Rightarrow -x + 4\, y$$

Hence, we get $(\partial K / \partial y) = 4\, y$

Integrating the above expression, we get $C = 4\, y^2/2 = 2\, y^2$

Substituting the value of K in Eq. 6, we get the general expression as

$$\psi = - y\, x - 2\, x^2 + 2\, y^2$$

July/Aug 2003

7. The components of velocity for a two dimensional flow are given by

$u = x\, y;$ $v = x2 - y2/2$

Check whether (i) they represent the possible case of flow and

(ii) the flow is irrotational. (6)

Solution:

$u = x\, y;$ and $v = x2 - y2/2$

$(\partial u / \partial x) = y$ $(\partial v / \partial y) = -y$

$(\partial u / \partial y) = x$ $(\partial v / \partial x) = 2x$

For a possible case of flow the velocity components should satisfy the equation of continuity.

i.e. $[(\partial u/\partial x)+(\partial v/\partial y)] = 0$

Substituting, we get $y - y = 0.$

Hence it is a possible case of a fluid flow.

For flow to be irrotational in a two dimensional fluid flow, the rotational component in z direction (ωz) must be zero, where

$$\omega_z = ½\, [(\partial v/\partial x) - (\partial u/\partial y)] = ½\, (2x - x) \neq 0$$

Hence, the flow is not irrotational.

July/Aug 2003

8. Find the components of velocity along x and y for the velocity potential $\phi = a$ Cos xy.

Also calculate the corresponding stream function. (8)

Solution:

$\phi = a$ Cos xy.

$(\partial \phi / \partial x) = -u = - ay$ Sin xy (1)

and $(\partial \phi / \partial y) = -v = - ax$ Sin xy (2)

Hence $u = ay$ Sin xy and $v = ax$ Sin xy.

Jan/Feb 2004

9. The stream function and velocity potential for a flow are given by

$\psi = 2xy$ and $\phi = x^2 - y^2$

Show that the conditions for continuity and irrotational flow are satisfied (8)

Solution:

From the properties of Stream function, the existence of stream function shows the possible case of flow and if it satisfies Laplace equation, then the flow is irrotational.

(i) $\psi = 2xy$

$(\partial \psi/\partial x) = 2y$ and $(\partial \psi/\partial y) = 2x$

$(\partial^2 \psi/\partial x^2) = 0$ and $(\partial^2 \psi/\partial y^2) = 0$

$(\partial^2 \psi/\partial x\, \partial y) = 2$ and $(\partial^2 \psi/\partial y\, \partial x) = 2$

$(\partial^2 \psi/\partial x\, \partial y) = (\partial^2 \psi/\partial y\, \partial x)$

Hence the flow is Continuous.

$(\partial^2 \psi/\partial x^2) + (\partial^2 \psi/\partial y^2) = 0$

As it satisfies the Laplace equation, the flow is irrotational.

From the properties of Velocity potential,

the existence of Velocity potential shows the flow is irrotational and if it satisfies Laplace equation, then it is a possible case of flow

(ii) $\phi = x^2 - y^2$

$(\partial \phi/\partial x) = 2x$ and $(\partial \phi/\partial y) = -2y$

$(\partial^2 \phi/\partial x^2) = 2$ and $(\partial^2 \phi/\partial y^2) = -2$

$(\partial^2 \phi/\partial x\, \partial y) = 0$ and $(\partial^2 \phi/\partial y\, \partial x) = 0$

$(\partial^2 \phi/\partial x\, \partial y) = (\partial^2 \phi/\partial y\, \partial x)$

Hence the flow is irrotational

$(\partial^2 \phi/\partial x^2) + (\partial^2 \phi/\partial y^2) = 0$

As it satisfies the Laplace equation, the flow is Continuous.

10. In a 2-D flow, the velocity components are $u = 4y$ and $v = -4x$

i. is the flow possible ?
ii. if so, determine the stream function
iii. What is the pattern of stream lines ?

Solution:

For a possible case of fluid flow, it has to satisfy continuity equation.

i.e. $[(\partial u/\partial x)+(\partial v/\partial y)] = 0$ (1)

$u = 4y$ and $v = -4x$

$(\partial u/\partial x) = 0$ $(\partial v/\partial y) = 0$

Substituting in Eq. 1, we get ***0.***

Hence the flow is possible.

Stream function

We know that $(\partial \psi/\partial x) = v = -4x$ (2)

and $(\partial \psi/\partial y) = -u = -4y$ (3)

$\psi = -2x^2 + C(y)$ (4)

Where ***C*** is an integral constant and a function of ***y.***

Differentiating Eq. 4, w.r.t. ***y***, we get

$(\partial \psi/\partial y) = 0 + \partial C(y)/\partial y = -u = -4y$

Integrating the above expression w.r.t. ***y*** we get

$C(y) = -2y^2.$

Substituting the above value in Eq. 4, we get the general expression as

$\psi = -2x^2 - 2y^2 = -2(x^2 + y^2)$

The above equation is an expression of concentric circles and hence the stream function is concentric circles.

11. A 250 mm diameter pipe carries oil of specific gravity 0.9 at a velocity of 3 m/s. At another section the diameter is 200 mm. Find the velocity at this section and the mass rate of flow of oil. (Jan/Feb 2005)

Solution:

$D_1 = 0.25\ m;\ D_2 = 0.2\ m;\ S_o = 0.9;\ V_1 = 3\ m/s;\ \rho = 1000\ kg/m^3;\ V2 = ?;$ Mass rate of flow = ?

From discharge continuity equation for steady incompressible flow, we have

$Q = A_1V_1 = A_2V_2$ (01)

$A_1 = (\pi/4)D_1^2 = (\pi/4)0.25^2 = 0.0499\ m^2$

$A_2 = (\pi/4)D_2^2 = (\pi/4)0.20^2 = 0.0314\ m^2$

Substituting in Eq. 1, we get

$Q = 0.0499 \times 3 = 0.1473\ m^3/s$

Mass rate of flow = $\rho Q = 0.1479 \times 1000 = 147.9\ kg/m^3$ *(Ans)*

$V_2 = (A_1 / A_2) \times V_1 = (D_1 / D_2)^2 \times V_1 = (0.25/0.2)^2 \times 3 = 4.6875\ m/s$ *(Ans)*

12. A stream function in a two dimensional flow is $\psi = 2xy$. Show that the flow is irrotational and determine the corresponding velocity potential. (8)

Solution:

For first part see Prob. 9.

Given $\psi = 2xy$.

$u = -(\partial\phi/\partial x) = -(\partial\psi/\partial y) = -2x$ (01)

$v = -(\partial\phi/\partial y) = (\partial\psi/\partial x) = 2y$ (02)

Integrating Eq. 1, w.r.t. x, we get

$\phi = 2x^2/2 + C = x^2 + C(y)$ (03)

Where $C(y)$ is an integral constant independent of x

Differentiating Eq. 3 w.r.t. y, we get

$(\partial\phi/\partial y) = 0 + (\partial C(y)/\partial y) = -2y$

Integrating the above expression w.r.t. y, we get

$C(y) = -y^2$

Substituting for $C(y)$ in Eq. 3, we get the general expression for ϕ as

$\phi = x^2 + C = x^2 - y^2$ *(Ans)*

13. The velocity potential for a flow is given by the function $\phi = x^2 - y^2$. Verify that the flow is incompressible and determine the stream function. (Aug 05) (10)

Solution:

From the properties of velocity potential, we have that if ϕ satisfies Laplace equation, then the flow is steady incompressible continuous fluid flow.

Given $\phi = x^2 - y^2$

$(\partial\phi/\partial x) = 2x$ $\quad$ $(\partial\phi/\partial y) = -2y$

$(\partial^2\phi/\partial x^2) = 2$ $\quad$ $(\partial^2\phi/\partial^2 y) = -2$

From Laplace Equation, we have $(\partial^2\phi/\partial x^2) + (\partial^2\phi/\partial^2 y) = 2 - 2 = 0.$

Finding out the stream function for the above velocity potential is reverse procedure of Prob. 12 and the answer is $\psi = 2xy$.

14. Find the corresponding stream function of a flow with velocity potential defined as $\boldsymbol{\phi = x(2y-1)}$ (Feb 2006) *(08)*

Solution:

Given $\phi = x(2y-1)$

Let ψ be the corresponding stream function

From the relationship between velocity potential and stream function we have

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}; \qquad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

Hence

$$\frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} = 2y - 1 \text{and} \qquad (01)$$

$$\frac{\partial \psi}{\partial x} = -\frac{\partial \phi}{\partial y} = -2x \qquad (02)$$

Integrating Eq. 01 with respect to y, we get

$$\int d\psi = \int (2y-1)\,dy \text{ or}$$

$$\psi = y^2 - y + C(x) \qquad (03)$$

Where $C(x)$ is an integral constant and may be a function of x

Differentiating the above expression with x, we get

$$\frac{\partial \psi}{\partial x} = \frac{\partial C(x)}{\partial x}$$

But from Eq. 02, we have $\frac{\partial \psi}{\partial x} = -2x = \frac{\partial C(x)}{\partial y}$

Integrating the above equation with respect to x, we get

$C(x) = -x^2$.

Substituting in Eq. 03, we get the general expression for the stream function as

$$\psi = y^2 - y - x^2$$

15. For the velocity components in a fluid flow given by $u=2xy$ and $v=x^2-y^2$, show that the flow is possible. Obtain the relevant stream function. (July 2006) (10)

Solution:

For the flow to be possible, it should satisfy the continuity equation given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (01)$$

$$\frac{\partial u}{\partial x} = 2y$$

$$\frac{\partial v}{\partial y} = -2y$$

Substituting in Eq. 01, we get $2y - 2y = 0$

Hence it satisfies continuity equation and the flow is continuous and Stream function exists.

Stream Function

Let ψ be the velocity potential.

Then $\boldsymbol{(\partial \psi/\partial x) = v = (x^2-y^2)}$ (02)

and $\boldsymbol{(\partial \psi/\partial y) = - u = -(2xy)}$ (03)

Integrating Eq. 3, with respect to *y*, we get

$$\psi = -xy^2 + K(x) \qquad (04)$$

Where ***K(x)*** is an integral constant, which is independent of ***y*** and can be a function of ***x*.**

Differentiating Eq. 5 w.r.t. ***x***, and from Eq. 2, we get

$$(\partial \psi / \partial x) = -y^2 + (\partial K(x) / \partial x) \Rightarrow (x^2 - y^2)$$

Hence, we get $(\partial K(x) / \partial x) = x^2$

Integrating the above expression, we get $K(x) = x^3/3$

Substituting the value of ***K(x)*** in Eq. 5, we get the general expression as

$$\psi = x^3/3 - x\,y$$